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Citation	数理解析研究所講究録 (1997), 1006: 22-46
Issue Date	1997-08
URL	http://hdl.handle.net/2433/61472
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
1006巻 1997年 22-46

THE MODULI SPACE OF ONCE PUNCTURED ELLIPTIC CURVES WITH LAGRANGIAN SUBLATTICES

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1. Introduction

We introduced a new method of constructing once punctured Riemann surfaces in [H-O1] (see also [H-O2]). In our construction we use line segments in the complex plane $\mathbb{C}$ and parallel transformations: For a pair of disjoint parallel line segments with the same length in $\mathbb{C}$, we first cut $\mathbb{C}$ along the segments and paste each side of one segment and the opposite side of the other segment by a parallel transformation obtaining a once punctured elliptic curve. The puncture is at infinity. We shall call such a pair an *Igeta.* (Igeta is a Japanese word coming from a technical term "Igeta-kuzushi" used in a Japanese martial art.) Putting g disjoint pieces of Igeta on $\mathbb{C}$, we obtain a once punctured Riemann surface of genus g in the same way. (See Figure 1. The numbers (1),...,(6) in Figure 1 indicate where to paste.) We denote a set of g disjoint Igeta by Γ and the resulting once punctured Riemann surface by $(R(\Gamma),p_{\infty})$. Moreover when we move the position of g Igeta, there appears a family of once punctured Riemann surfaces of genus g. All the possible configurations of g disjoint Igeta up to the affine automorphisms of $\mathbb{C}$ form a $3g-2$-dimensional complex V-manifold and this dimension is the same as the dimension of the moduli space $\mathcal{M}_{g,1}$ of once punctured Riemann surfaces of genus g. We thus expect to have a visual image of the moduli space by using this construction.

Let $I_g\eta_0$ be the collection of those Γ having $[0,1]$ as one of its $2g$ line segments. $I_g\eta_0$ turns out to be a $3g-2$-dimensional complex manifold. We showed in [H-O1] that the Kodaira-Spencer map

$$\rho_{\Gamma,0}\ : T(I_g\eta_0)_\Gamma \longrightarrow H^1(R(\Gamma);\Theta(-p_\infty))$$

is an isomorphism for any $\Gamma\in I_g\eta_0$, where $T(I_g\eta_0)_\Gamma$ is the holomorphic tangent space of $I_g\eta_0$ at Γ and $\Theta(-p_\infty)$ is the sheaf of germs of holomorphic vector fields on $R(\Gamma)$ having zero at p_∞. This implies that the family of once punctured Riemann surfaces of genus g by Igeta-construction is complete and effectively parametrized at any point for each g.

We also showed that any once punctured Riemann surface (R,p) can be obtained from $\mathbb{C}$ by cutting along line segments and pasting by parallel transformations. (Note that Igeta-construction is a special way of cutting and pasting.) This result is obtained by considering a *Lagrangian sublattice* Λ of R, a subgroup of $H_1(R;\mathbb{Z})$ which coincides its orthogonal complement with respect to the intersection form on $H_1(R;\mathbb{Z})$.

When we construct a once punctured Riemann surface $(R(\Gamma),p_\infty)$ from Γ, $R(\Gamma)$ has a natural Lagrangian sublattice Λ_Γ. Igeta-construction leads us to consider the moduli space of once punctured Riemann surfaces with Lagrangian sublattices.

In this paper we consider the case of genus 1, and describe the moduli space using a natural extension of Igeta-construction, that is, we make a complete list of once puctured elliptic curves with Lagrangian sublattices (see §2):

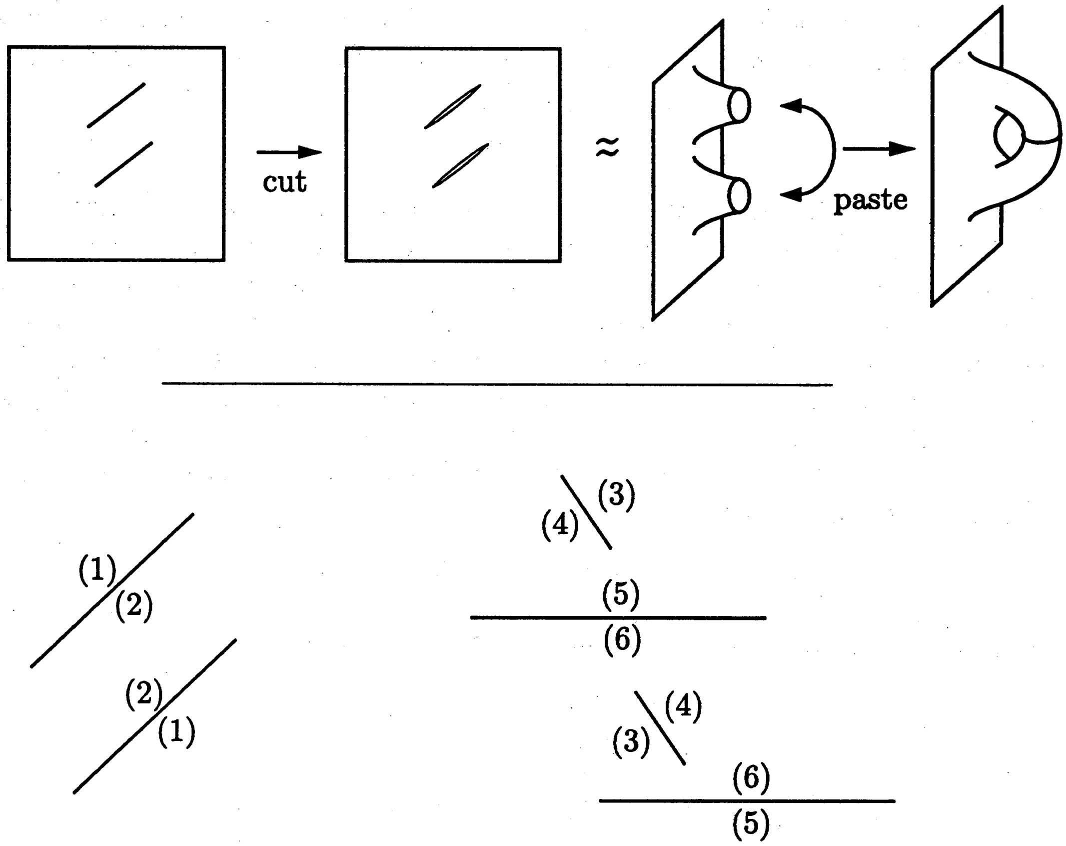


FIGURE 1. Igeta-construction

Theorem 1 . *For any once punctured elliptic curve with a Lagrangian sublattice* (E, p, Λ), *there exists one and only one* $(E(a,b,x), p_\infty, \Lambda_0)$ *isomorphic to* (E, p, Λ).

For any once punctured Riemann surface (R, p), a Lagrangian sublattice Λ and the puncture p determine a certain Abelian differential ω_Λ of the second kind up to scalars. To prove Theorem 1 we study the geometry of geodesics on once punctured elliptic curves having metrics with conical singularities induced by the Abelian differentials ω_Λ. For each $(E(a,b,x), p_\infty, \Lambda_0)$, all the closed geodesics can be described visually by using our construction.

In §3 we consider the complex structure of the moduli space of once punctured elliptic curves with Lagrangian sublattices by using the description given in §2.

The authors would like to thank the many people who have contributed ideas and suggestions for this manuscript, among them C. F. Bödigheimer, V. Chueshev, K. Fukaya, M. Furuta, A. Hattori, S. Morita and K. Ono. The authors would like to express their gratitude to H. Helling for useful suggestions on how to improve the early drafts.

2. ELLIPTIC CURVES WITH LAGRANGIAN SUBLATTICES

In this section, we give a description of all the once punctured elliptic curves with Lagrangian sublattices by using a natural extension of Igeta-construction.

We prepare some notations and terms about the metrics induced by Abelian differentials. For an Abelian differential ω on a closed Riemann surface R, we denote by $\omega^{-1}(0)$

(resp. $\omega^{-1}(\infty)$) the set of zeros (resp. poles) of ω. We call a simple path or simple loop $\gamma : [0,1] \to R$ *ω-line-segment* if its image contains no poles of ω and the integral

$$\int_{\gamma(0)}^{\gamma(t)} \omega$$

depends on $t \in [0,1]$ linearly. We also call its image *ω-line-segment.* Let us denote by g_ω the flat metric on $R - \omega^{-1}(\infty)$ induced by the 2-form $\frac{i}{2}\omega \wedge \bar{\omega}$, which has conical singularities at $\omega^{-1}(0)$. Then ω-line-segments are geodesics for g_ω. Let us call a ω-line-segment $\gamma : [0,1] \to R - \omega^{-1}(\infty)$ *ω-edge* if $\gamma([0,1]) \cap \omega^{-1}(0) = \{\gamma(0), \gamma(1)\}$. Then closed geodesics with respect to g_ω which contain some zeros of ω consist of ω-edges because the integral of ω gives rise to a local isometry between $R - (\omega^{-1}(\infty) \cup \omega^{-1}(0))$ and $\mathbb{C}$. An Abelian differential ω of the second kind on a closed Riemann surface R induces an element of $H^1(R;\mathbb{C})$, and further an element denoted by $\mathrm{PD}[\omega]$ of $H_1(R;\mathbb{C})$ via the Poincaré duality. It holds that

$$\int_\alpha \omega = \alpha \cdot \mathrm{PD}[\omega] \quad \text{for any } \alpha \in H_1(R;\mathbb{Z}).$$

We look for a singular 1-cycle σ representing $\mathrm{PD}[\omega]$ such that

$$\sigma = \sum_{k=1}^{N} c_k \gamma_k, \quad c_k \in \mathbb{C}$$

where $\gamma_1, \ldots, \gamma_N$ are ω-edges. (We can find it if $\omega^{-1}(0)$ is nonempty.)

Let (R,p) be a once punctured Riemann surface of genus g and Λ a Lagrangian sublattice of $H_1(R;\mathbb{Z})$. The kernel Z_Λ of the homomorphism given by Abelian integrals

$$H^0(R;\Omega^1(2p)) \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{C}) \ (\cong \mathbb{C}^g)$$

is always one-dimensional because it holds that

$$\dim H^0(R;\Omega^1(2p)) = g+1$$

from the Riemann-Roch formula and the surjectivity is implied by the bilinear relations of Riemann. Accordingly, a Lagrangian sublattice and a point on the surface determine an Abelian differential up to scalars. (Note that $\Lambda \cong \mathbb{Z}^g$.)

From now on we consider the case of genus one.

Figure 2 is a list of some once punctured elliptic curves with Lagrangian sublattices denoted by $\widetilde{E}(a,b,x)$ (or $(E(a,b,x), p_\infty, \Lambda_0)$) constructed from $\mathbb{C}$ by cutting along the line segments and pasting by parallel transformations. (The numbers (1), (2), ... in each figure indicate pasting data or where to paste.) They split into the following four types, and each once punctured elliptic curve $(E(a,b,x), p_\infty)$ is constructed as follows:

I: (The case where $a = 0$, $b = 1$, and x is a complex number in the upper half plane $\mathbf{H}$ or a real number in the interval $(0,1)$.) Cut the complex plane along

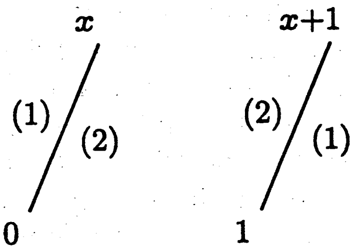
$$x([0,1]) \cup (x([0,1]) + 1),$$

and paste each side of $x([0,1])$ and the opposite side of $(x([0,1]) + 1)$ by a parallel transformation. (Igeta-construction)

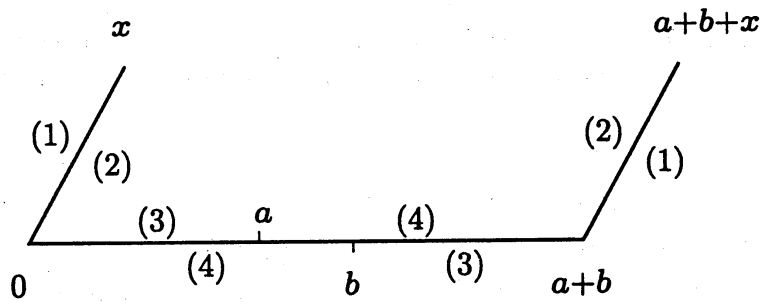
II: (The case where a and b are relatively prime positive integers, and x is a complex number in the upper half plane $\mathbf{H}$.) Cut the complex plane along

$$x([0,1]) \cup [0, a+b] \cup (x([0,1]) + a + b),$$

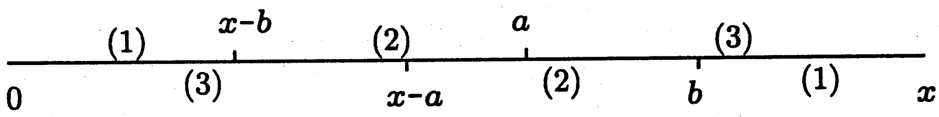
I $(E(0,1,x), p_\infty)$ where $x \in \mathbf{H} \cup (0,1)$.



II $(E(a,b,x), p_\infty)$ where $a,b \in \mathbb{Z}+$, $(a,b)=1$, $x \in \mathbf{H}$.



III $(E(a,b,x), p_\infty)$ where $a,b \in \mathbb{Z}+$, $a \neq b$, $(a,b)=1$, $x \in (\max(a,b), a+b)$.



IV $(\mathrm{E}(a,b,0), p_\infty)$ where $a,b \in \mathbb{Z}+$, $(a,b)=1$.

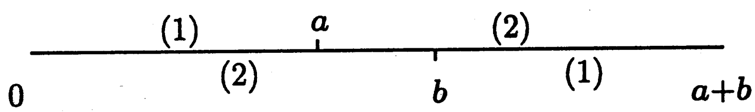


FIGURE 2. The list of $\widetilde{E}(a,b,x)$'s

and paste each side of $x([0,1])$ and the opposite side of $(x([0,1])+a+b)$ by a parallel transformation and paste the upper side of $[0,a]$ (resp. $[a,a+b]$) and the lower side of $[b,a+b]$ (resp. $[0,b]$) by a parallel transformation.

III: (The case where a and b are distinct relatively prime positive integers, and x is a real number such that $x \in (\max(a,b), a+b)$.) Cut the complex plane along $[0,x]$, and paste the lower side of $[0,x-a]$ (resp. $[x-a,b]$, $[b,x]$) and the upper side of $[a,x]$ (resp. $[x-b,a]$, $[0,x-b]$) by a parallel transformation.

IV: (The case where a and b are relatively prime positive integers, and $x=0$.) Cut the complex plane along $[0,a+b]$ and paste the upper side of $[0,a]$ (resp. $[a,a+b]$) and the lower side of $[b,a+b]$ (resp. $[0,b]$) by a parallel transformation.

Note that each elliptic curve constructed in this way has a natural Abelian differential ω_0 induced by the differential $d\zeta$ of the standard coordinate ζ of $\mathbb{C}$ and that the Lagrangian sublattice Λ_0 of $\widetilde{E}(a,b,x)$ is characterized as the kernel of the period map of ω_0 from $H_1(E(a,b,x);\mathbb{Z})$ to $\mathbb{C}$. Further, the primitive period of ω_0 is equal to 1 because a and b are relatively prime. When $\widetilde{E}(a,b,x)$ is of type I, II, or III, the set $\omega_0^{-1}(0)$ consists of the two points p_0 and p_1 coming from the origin and the point x of $\mathbb{C}$, and when $\widetilde{E}(a,b,x)$ is of type IV, the set $\omega_0^{-1}(0)$ consists of the one point p_0 coming from the origin in $\mathbb{C}$. (See Figure 2.)

Our goal in this section is to prove

Theorem 1 . *For any once punctured elliptic curve with a Lagrangian sublattice (E,p,Λ), there exists one and only one $\widetilde{E}(a,b,x)$ isomorphic to (E,p,Λ).*

We call two once punctured Riemann surfaces with Lagrangian sublattices (R,p,Λ) and (R',p',Λ') *isomorphic*, if there exists a biholomorphic map from R to R' transforming p to p' and Λ into Λ'.

In order to prove Theorem 1 we investigate the geometry of geodesics on once punctured elliptic curves having metrics with conical singularities induced by Abelian differentials.

For a once punctured elliptic curve with a Lagrangian sublattice (E,p,Λ), there exists an Abelian differential ω_Λ, unique up to sign, in the kernel Z_Λ (see [H-O1] or [H-O2]) such that the primitive period of ω_Λ is equal to 1, because Z_Λ is one-dimensional and any non-trivial element in Z_Λ has non-trivial periods. Then $\mathrm{PD}[\omega_\Lambda]$ is an integral homology class. Since the metric g_{ω_Λ} on $E-p$ depends only on Λ in this case, we shall shortly denote this metric by g_Λ. For the same reason we shall also use the term Λ-*edge* instead of ω_Λ-edge.

We first show that $\widetilde{E}(a,b,x)$'s are not isomorphic to one another by studying Λ_0-edges.

For each $\widetilde{E}(a,b,x)$, the Abelian differential ω_{Λ_0} coincides with ω_0 up to sign. So Λ_0-edges are line segments in $\mathbb{C}$ between the points of $\omega_0^{-1}(0)$ in each case of Figure 2. Accordingly we can easily obtain the following proposition.

Proposition 1 . *All the Λ_0-edges can be described as in Figure 3 for each $\widetilde{E}(a,b,x)$.*

Proposition 1 implies that all the closed geodesics containing some zeros of ω_0, which consist of Λ_0-edges, can be described visually for each $\widetilde{E}(a,b,x)$. This proposition yields the following corollary.

Corollary 1 . *If $(a,b,x) \neq (a',b',x')$, then $\widetilde{E}(a,b,x)$ is not isomorphic to $\widetilde{E}(a',b',x')$.*

Proof. Suppose (E,p_∞,Λ_0) is isomorphic to $\widetilde{E}(a,b,x)$. We shall recover the data (a,b,x) from (E,p_∞,Λ_0).

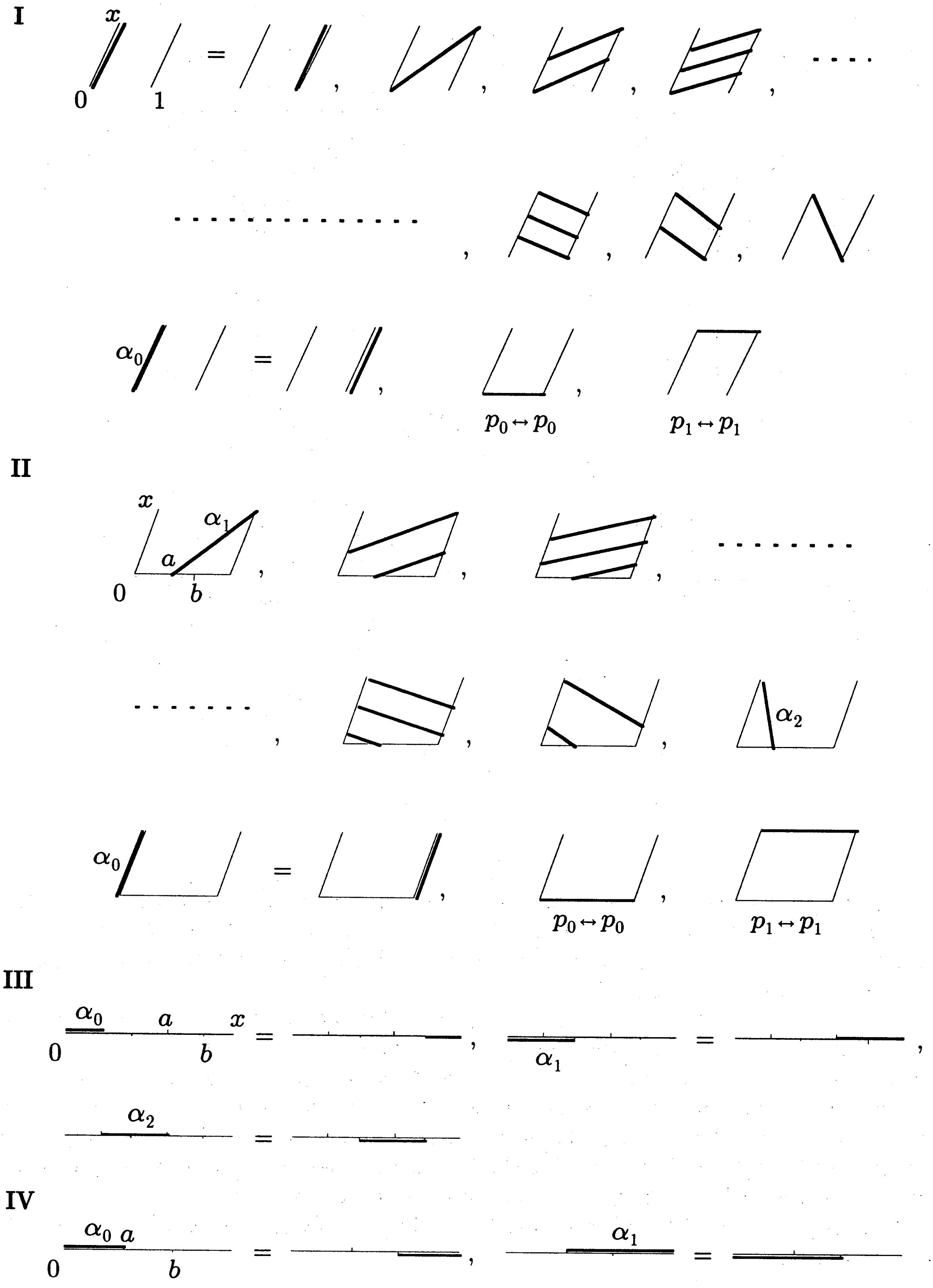


FIGURE 3. Λ_0-edges

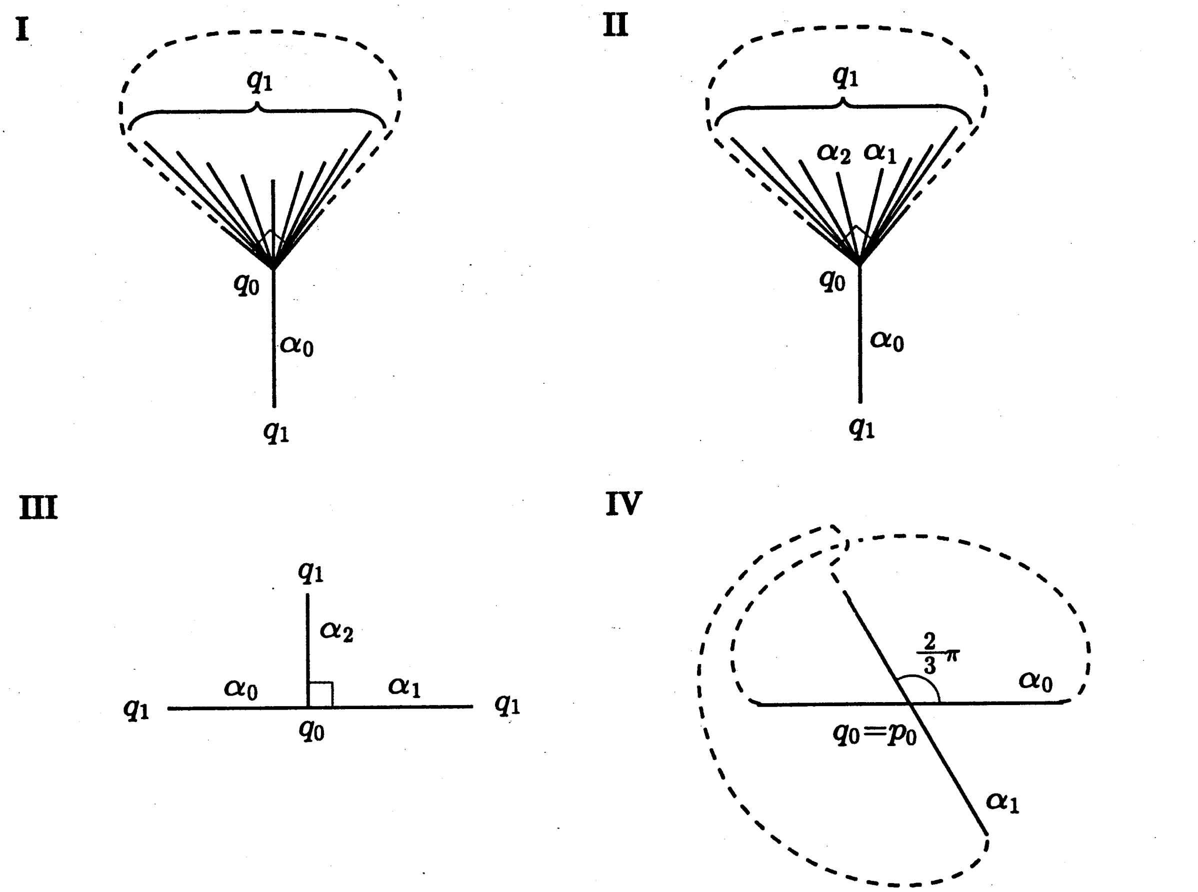


FIGURE 4

We distinguish the type IV from others by the degeneracy of zeros of ω_{Λ_0}, and distinguish the types I, II and III from each other by the properties of Λ_0-edges on E. Let $\{q_0, q_1\}$ be the set of singular points of g_{Λ_0}. (When (E, p_∞, Λ_0) is of type IV, let q_0 be the unique singular point of g_{Λ_0}.)

When we describe all the directions of Λ_0-edges around q_0, we obtain Figure 4 from Proposition 1. We can specify the Λ_0-edges α_0, α_1 and α_2 on E in Figure 4.

Note that these figures do not depend on the choice of q_0, and that there exists a Λ_0-edge which has the opposite direction of α_0 in the case where (E, p_∞, Λ_0) is of type I, however in the case where (E, p_∞, Λ_0) is of type II there exists no Λ-edge of this kind. We thus specify the type of (E, p_∞, Λ). We shall orient the Λ_0-edges α_0, α_1 and α_2 in Figure 4, which are also in Figure 3, from q_0 to q_1 in the case of type I, II or III.

When (E, p_∞, Λ_0) is of type I or II, we choose the sign of ω_{Λ_0} such that $\mathrm{Im} \int_{\alpha_0} \omega_{\Lambda_0} > 0$. Then we obtain

$$x = \int_{\alpha_0} \omega_{\Lambda_0}.$$

In case (E, p_∞, Λ_0) is of type II, we also obtain the following: (See Figure 3)

$$a = \int_{\alpha_0 - \alpha_2} \omega_{\Lambda_0},$$
$$b = \int_{\alpha_1 - \alpha_0} \omega_{\Lambda_0}.$$

When (E, p_∞, Λ_0) is of type III, we choose the sign of ω_{Λ_0} such that $\int_{\alpha_0} \omega_{\Lambda_0} > 0$. Then we obtain the following: (See Figure 3)

$$a = \int_{\alpha_0 - \alpha_2} \omega_{\Lambda_0},$$
$$b = \int_{\alpha_1 - \alpha_2} \omega_{\Lambda_0},$$
$$x = \int_{\alpha_0 + \alpha_1 - \alpha_2} \omega_{\Lambda_0}.$$

When (E, p_∞, Λ_0) is of type IV, we obtain the following: (See Figure 3)

$$a = \left| \int_{\alpha_0} \omega_{\Lambda_0} \right|,$$
$$b = \left| \int_{\alpha_1} \omega_{\Lambda_0} \right|.$$

We thereby recover the data (a, b, x) from (E, p_∞, Λ_0) in any case. Hence this corollary follows. □

Corollary 1 implies the uniqueness in Theorem 1. In order to finish our proof of Theorem1, we next show the completeness; for a given once punctured elliptic curve with a Lagrangian sublattice (E, p, Λ), there exists an $\widetilde{E}(a, b, x)$ which is isomorphic to (E, p, Λ).

Let γ be an oriented loop or map from S^1 to E representing a generator of Λ. The Abelian differential ω_Λ is exact on $E - \gamma(S^1)$ because the integral of ω_Λ along a loop α equals zero if and only if α represents a homology class whose intersection number with the homology class $[\gamma]$ equals zero. When we denote by $\|\gamma\|$ the length of γ with respect to the metric g_Λ, it is verified from the following two facts (1), (2) that there exists a loop representing a generator of Λ which has the smallest length among such loops, and that the loop consists of Λ-edges. We shall call a loop consisting of Λ-edges a *Λ-edge-loop.*

(1): For any loop α on E, there exists a Λ-edge-loop α' consisting of Λ-edges such that α' is homotopic to α and $\|\alpha'\| \leq \|\alpha\|$.

(2): For any positive number r, there exist at most finitely many Λ-edge-loops with length smaller than r.

We can show that fact (1) follows this way: we first approximate α by a polygonal loop, and then we reduce the number of vertices which do not lie on $\omega_\Lambda(0)$. On the other hand, for a positive number r, it is obvious that there exist at most finitely many Λ-edges with length smaller than r, and then the fact (2) follows.

Lemma 1. *The image of an oriented Λ-edge-loop γ representing a generator of Λ contains $\omega_\Lambda^{-1}(0)$.*

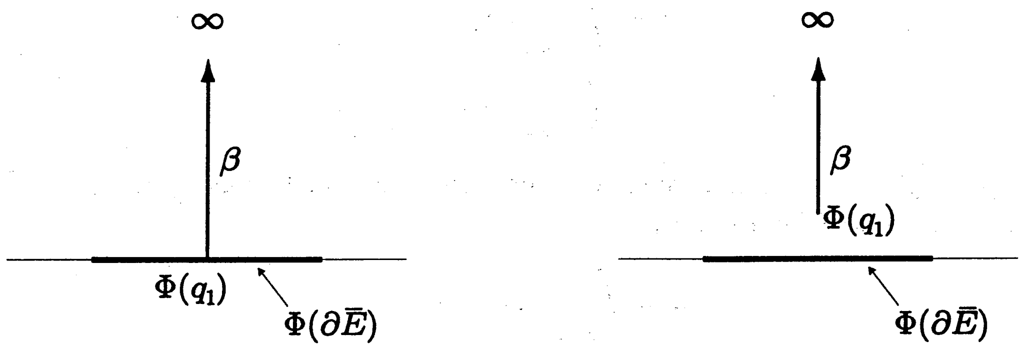


FIGURE 5

Proof. The Abelian differential ω_Λ has one point as zero of order 2, or has two points as zeros of order 1. We show the lemma in the second case. (In the other case the statement is trivial.) We set $\omega_\Lambda^{-1}(0) = \{q_0, q_1\}$.

Suppose the image of γ contains q_0 and does not contain q_1. (The image of γ consists of only Λ-edges from q_0 to q_0 itself.)

We cut along γ with E and denote by $\bar{E}$ the resulting Riemann polygon or Riemann surface with line-segment-boundary. (See [H-O1] or [H-O2].) Let $\tilde{q}_0$ be a point in $\bar{E}$ corresponding to q_0. The holomorphic map

$$\Phi(q) = \int_{q_0}^{q} \omega_\Lambda$$

from $E-(\gamma(S^1)\cup\{p\})$ to $\mathbb{C}$ extends to one from $\bar{E}$ to $\mathbb{C}P_1(=\mathbb{C}\cup\{p_\infty\})$, and Φ maps a local neighborhood of any point which is not in $\omega_\Lambda^{-1}(0)$ to a subset of $\mathbb{C}P_1$ biholomorphically. Moreover, Φ maps the boundary $\partial\bar{E}$ of $\bar{E}$ into the real axis in $\mathbb{C}P_1$, because ω_Λ only has real periods.

Then we shall find two paths from q_1 to p

$$\alpha_1, \alpha_2 : \; [0,1] \longrightarrow \bar{E}$$

such that

$$\alpha_1([0,1]) \cap \alpha_2([0,1]) = \{q_1, p\},$$
$$(\Phi\circ\alpha_1)([0,1]) = (\Phi\circ\alpha_2)([0,1]).$$

We fix a path β on $\mathbb{C}P_1$ from $\Phi(q_1)$ to p_∞ such that $\beta([0,1]) \cap \Phi(\partial\bar{E}) \subseteq \{\Phi(q_1)\}$. (See Figure 5.) Around q_1 we can take two distinct paths

$$\alpha_1', \alpha_2' : \; [0,\epsilon] \longrightarrow \bar{E}$$

such that

$$\alpha_1'([0,1]) \cap \alpha_2'([0,1]) = \{q_1\},$$
$$(\Phi\circ\alpha_1')(t) = (\Phi\circ\alpha_2')(t) = \beta(t) \quad (t \in [0,\epsilon]).$$

Since Φ is a local biholomorphic map except q_0, q_1, we obtain the desired paths α_1, α_2 as the pull-back of β by Φ which coincide near q_1 with α_1', α_2' respectively. Meanwhile, the map Φ is biholomorphic around p. This is a contradiction. $\square$

We recall three well-known facts about simple loops on elliptic curves.

Fact (1): The homology class represented by a simple loop is trivial or primitive.

Fact (2): If γ and γ' are simple loops such that their homology classes $[\gamma]$, $[\gamma']$ are primitive and their intersection number $[\gamma]\cdot[\gamma']$ is equal to zero, then $[\gamma]=\pm[\gamma']$.

Fact (3): If γ and γ' are simple loops such that their intersection number is equal to ± 1, then the pair $([\gamma],[\gamma'])$ is a basis of $H_1(E;\mathbb{Z})$.

Since the integral value of ω_Λ along any Λ-edge does not vanish, the following lemma follows from Fact (2) and the definition of Λ-edge immediately, and we omit its proof. (Note that two distinct parallel Λ-edges have no common points except the points in $\omega_\Lambda^{-1}(0)$.)

Lemma 2. *Suppose $\omega_\Lambda^{-1}(0)$ consists of two distinct points q_0, q_1. If γ_1 (resp. γ_2) is a Λ-edge from q_0 to q_0 itself (resp. q_1 to q_1 itself), then $[\gamma_1]=\pm[\gamma_2]\in H_1(E;\mathbb{Z})$.*

Let $\gamma_{(0)}$ be an oriented Λ-edge-loop representing a generator of Λ and having the smallest length. When we consider $\gamma_{(0)}$ as an ordered set of oriented Λ-edges, we may obtain another loop $\gamma_{(0)}'$ representing the same homology class by reordering the oriented Λ-edges suitably. (We assume that each Λ-edge inherits its orientation from $\gamma_{(0)}$.) We obtain the following lemma about $\gamma_{(0)}$.

Lemma 3. *The image of $\gamma_{(0)}$ consists of Λ-edges which have no intersection with one another except $\omega^{-1}(0)$.*

Proof. When ω_Λ has a zero of order 2, the lemma is immediate because all the Λ-edges are parallel. We thus consider the case where ω_Λ has two points q_0, q_1 as zeros.

We consider $\gamma_{(0)}$ as an ordered set $(\gamma_1,\gamma_2,\dots,\gamma_l)$ of oriented Λ-edges. Suppose Λ-edges γ_i and γ_j $(1\le i\le j\le l)$ have an intersection at a point $q\notin\omega_\Lambda^{-1}(0)$. Then both γ_i and γ_j are Λ-edges between q_0 and q_1, and they intersect each other transversely.

We prepare paths γ_i' and γ_j' modifying γ_i and γ_j around q as in Figure 6.

We replace γ_i, γ_j with γ_i', γ_j' respectively, and set

$$\gamma_{(0)}'=(\gamma_1,\dots,\gamma_i,\dots,\gamma_j,\dots,\gamma_l).$$

Then the cycle $\gamma_{(0)}'$ represents the homology class $[\gamma_{(0)}]$.

In the case (1) or (2), we can consider $\gamma_{(0)}'$ as a loop, and the length $\|\gamma_{(0)}'\|$ is smaller than $\|\gamma_{(0)}\|$. This contradicts the definition of $\gamma_{(0)}$.

In the case (3) or (4), it is necessary to reorder the elements of $\gamma_{(0)}'$. If there still exist Λ-edges between q_0 and q_1 in $\gamma_{(0)}'$, then it is easy to obtain a loop by reordering the paths and loops $\{\gamma_1,\dots,\gamma_i',\dots,\gamma_j',\dots,\gamma_k\}$ whose length is smaller than $\|\gamma_{(0)}\|$. This also contradicts the definition of $\gamma_{(0)}$. If there exist no more Λ-edges between q_0 and q_1 in $\gamma_{(0)}'$, we further replace the loop γ_i' by Λ-edges from q_0 to q_0 itself, the loop γ_j' by Λ-edges from q_1 to q_1 itself preserving the homology class. Moreover, by using Lemma 2, we can replace all the Λ-edges from q_1 to q_1 itself by Λ-edges from q_0 to q_0 preserving the homology class.

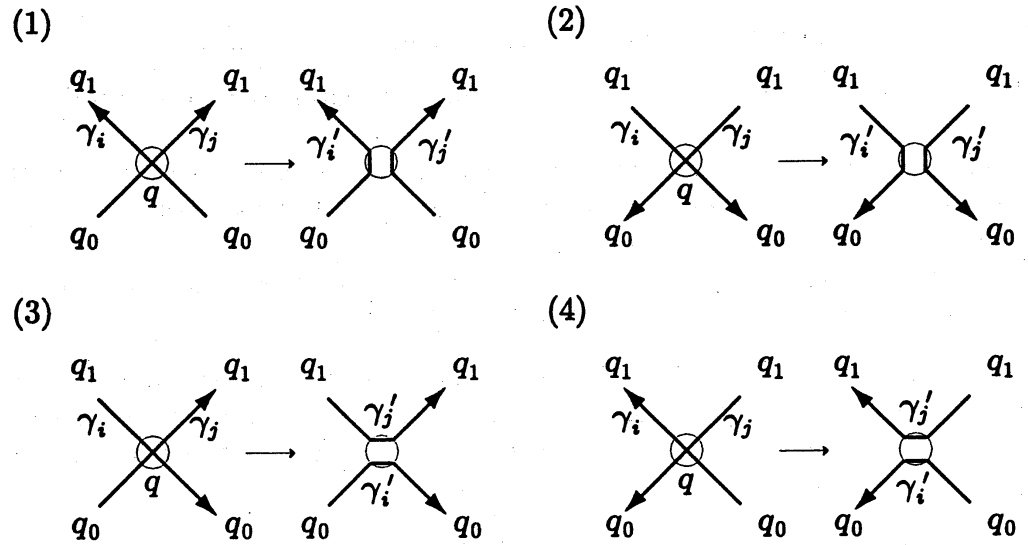


FIGURE 6

Then we obtain a cycle representing the cohomology class $[\gamma_{(0)}]$, and consisting of only Λ-edges from q_0 to q_0 itself. This cycle can be considered as a loop, and this contradicts Lemma 1. □

Proof of Theorem 3. Let $\gamma_{(0)}$ be an oriented Λ-edge-loop representing a generator of Λ and having the smallest length as before. We describe $\gamma_{(0)}$ as an ordered set of oriented Λ-edges;

$$\gamma_{(0)} = (\gamma_1, \ldots, \gamma_l).$$

We first consider the case where $\omega_\Lambda^{-1}(0)$ consists of only one point q_0. In this case, each element γ_k of $\gamma_{(0)}$ is an oriented simple loop on E, and the integral of ω_Λ along γ_k is a non-zero integer. Hence it follows from Fact (2) that γ_k represents a primitive class in $H_1(E;\mathbb{Z})$. We shall choose the sign of ω_Λ such that

$$\int_{\gamma_1} \omega_\Lambda > 0.$$

Since the integral of ω_Λ along $\gamma_{(0)}$ is equal to zero, we may further assume by reordering

$$\int_{\gamma_2} \omega_\Lambda < 0.$$

The loops γ_1 and $-\gamma_2$ represent different homology classes, because $\gamma_{(0)}$ has the smallest length. On the other hand, they intersect each other only at q_0. Hence from Facts (1) and (2) it follows that their intersection number at q_0 is equal to 1 or -1 and that the pair $([\gamma_1],[\gamma_2])$ is a basis of $H_1(E;\mathbb{Z})$.

Therefore we can describe the direction as in (i) or (ii) in Figure 7.

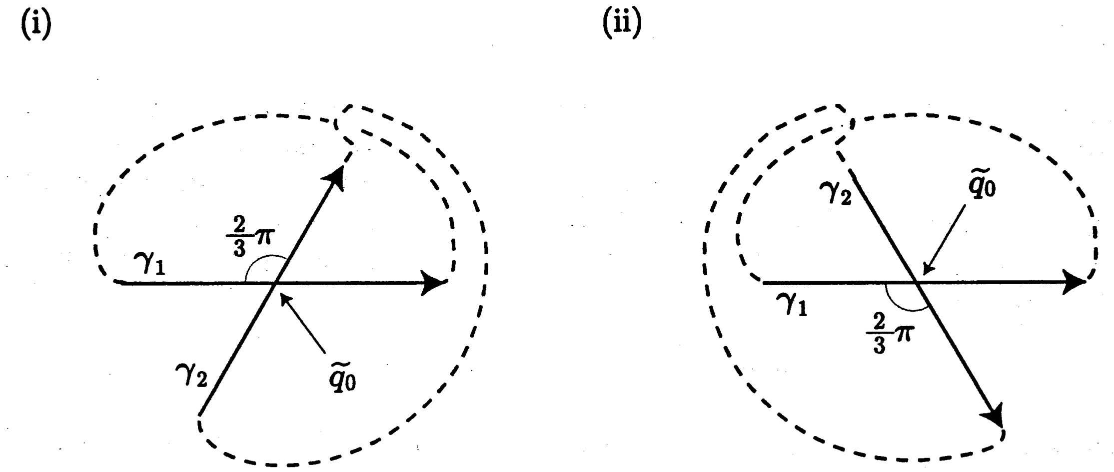


FIGURE 7

Cut E along γ_1 and γ_2, and denote by $\bar{E}$ the resulting Riemann polygon. In each case in Figure 7, by choosing a point $\tilde{q}_0$ in $\partial\bar{E}$ suitably (see Figure 7), we obtain a holomorphic map Φ from $\bar{E}$ to $\mathbb{C}P_1$:

$$\Phi(q) = \int_{\tilde{q}_0}^{q} \omega_\Lambda.$$

In case (i), set

$$a = \int_{\gamma_1} \omega_\Lambda,$$
$$b = -\int_{\gamma_2} \omega_\Lambda.$$

In case (ii), set

$$a = -\int_{\gamma_2} \omega_\Lambda,$$
$$b = \int_{\gamma_1} \omega_\Lambda.$$

Then the integers a and b are relatively prime because the pair $([\gamma_1], [\gamma_2])$ is a basis of $H_1(E;\mathbb{Z})$, and then Φ gives rise to an isomorphism between (E, p, Λ) and $\widetilde{E}(a, b, 0)$.

We next consider the other case: $\omega_\Lambda^{-1}(0)$ consists of two distinct points q_0 and q_1. In this case, from Lemma 1 there exists at least one Λ-edge from q_0 to q_1 and at least one Λ-edge from q_1 to q_0 in $\gamma_{(0)}$. We may assume that γ_1 is a Λ-edge from q_0 to q_1. We may further assume from Lemma 2 that $\gamma_{(0)}$ does not contain Λ-edges from q_1 to q_1 itself. Since γ_1 is a Λ-edge from q_0 to q_1, the Λ-edge γ_2 is from q_1 to q_0.

(i) Suppose there exists a Λ-edge γ_i from q_1 to q_0 in $\gamma_{(0)}$ such that

$$\int_{\gamma_i} \omega_\Lambda = -\int_{\gamma_1} \omega_\Lambda.$$

Because $\gamma_{(0)}$ has the smallest length, the simple loop (γ_1, γ_i) is non-trivial, and hence primitive (Fact (1)). Therefore we obtain $\gamma_{(0)} = (\gamma_1, \gamma_i)$. (Note that a loop γ representing a generator of Λ is characterized by two properties; one is that γ represents a primitive homology class, and the other is that the integral of ω_Λ along γ vanishes.)

Cut E along γ_1 and γ_i, denote by $\bar{E}$ the resulting Riemann polygon, and set the signature of ω_Λ such that

$$\mathrm{Im}(\int_{\gamma_1} \omega_\Lambda) > 0 \quad \text{or} \quad \int_{\gamma_1} \omega_\Lambda > 0.$$

By choosing a point $\tilde{q}_0$ in $\partial\bar{E}$ suitably, we obtain a holomorphic map Φ from $\bar{E}$ to $\mathbb{C}P_1$:

$$\Phi(q) = \int_{\tilde{q}_0}^{q} \omega_\Lambda.$$

If we set

$$x = \int_{\gamma_1} \omega_\Lambda,$$

then Φ gives rise to an isomorphism between (E, p, Λ) and $\widetilde{E}(0,1,x)$. (Igeta-construction)

(ii) Suppose

$$\int_{\gamma_i} \omega_\Lambda \neq -\int_{\gamma_1} \omega_\Lambda$$

for any γ_i, Λ-edge from q_1 to q_0 in $\gamma_{(0)}$.

(ii-a) When the integral of ω_Λ along γ_1 is not a real number, we choose the sign of ω_Λ such that

$$\mathrm{Im}(\int_{\gamma_1} \omega_\Lambda) > 0.$$

We may assume without loss of generality that there exists a Λ-edge γ_i from q_1 to q_0 in $\gamma_{(0)}$ such that the integral of ω_Λ along $\gamma_1 + \gamma_i$ is a positive integer. We may further assume by reordering that the integral of ω_Λ along $\gamma_1 + \gamma_2$ is minimal among such γ_i's.

$$\int_{\gamma_1+\gamma_i} \omega_\Lambda \geq \int_{\gamma_1+\gamma_2} \omega_\Lambda \geq 0$$

Since the integral of ω_Λ along $\gamma_{(0)}$ vanishes, we see from the inequality above that there occur the following two cases:

1: There exists in $\gamma_{(0)}$ a Λ-edge γ_j from q_0 to q_0 such that

$$\int_{\gamma_j} \omega_\Lambda < 0.$$

2: There exist in $\gamma_{(0)}$ Λ-edges γ_μ and γ_ν (γ_μ is from q_0 to q_1, γ_ν from q_1 to q_0) such that

$$\int_{\gamma_\mu+\gamma_\nu} \omega_\Lambda < 0.$$

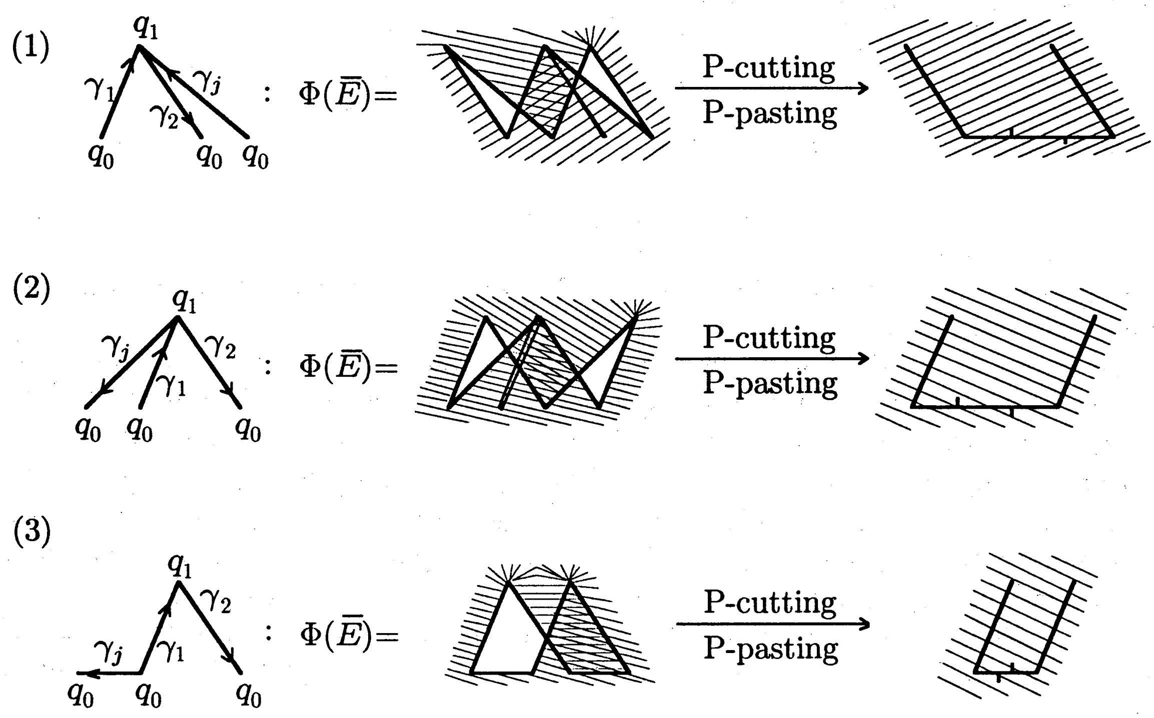


FIGURE 8

Suppose the following two inequalities hold for any γ_μ from q_0 to q_1 and for any γ_ν from q_1 to q_0.

$$\int_{\gamma_2+\gamma_\mu} \omega_\Lambda \geq 0, \quad \int_{\gamma_1+\gamma_\nu} \omega_\Lambda \geq 0$$

Then we obtain the following inequality from the second inequality above and the choice of γ_2.

$$\int_{\gamma_1+\gamma_\nu} \omega_\Lambda \geq \int_{\gamma_1+\gamma_2} \omega_\Lambda$$

Furthermore we obtain the following inequality, and this is a contradiction.

$$\begin{aligned}\int_{\gamma_\mu+\gamma_\nu} \omega_\Lambda &= \int_{\gamma_\mu+\gamma_2} \omega_\Lambda - \int_{\gamma_2+\gamma_1} \omega_\Lambda + \int_{\gamma_1+\gamma_\nu} \gamma_\Lambda \\ &\geq \int_{\gamma_\mu+\gamma_2} \omega_\Lambda \\ &\geq 0.\end{aligned}$$

Therefore the following three cases occur:
(1) there is γ_j from q_0 to q_1 such that

$$\int_{\gamma_2+\gamma_j} \omega_\Lambda < 0.$$

(2) there is γ_j from q_1 to q_0 such that

$$\int_{\gamma_1+\gamma_j} \omega_\Lambda < 0.$$

(3) there is γ_j from q_0 to q_0 such that

$$\int_{\gamma_j} \omega_\Lambda < 0.$$

The figures on the left-hand side of Figure 8 indicate the images of γ_1, γ_2, and γ_j by the integral of ω_Λ in each case above.

In the case (1), we shall show that the homology classes represented by the simple loops (γ_1, γ_2) and (γ_j, γ_2) form a basis of $H_1(E;\mathbb{Z})$; If the homology classes $[(\gamma_1, \gamma_2)]$ and $-[(\gamma_j, \gamma_2)]$ represent the same homology class, then we get a loop $\gamma'_{(0)}$ representing a generator of Λ by removing γ_1, γ_2, and γ_j from $\gamma_{(0)}$ and placing $-\gamma_2$ instead, and $\gamma'_{(0)}$ is shorter than $\gamma_{(0)}$. This is a contradiction. Hence from Facts (1) and (2) it follows that the intersection number $[(\gamma_1, \gamma_2)] \cdot [(\gamma_j, \gamma_2)]$ is not equal to zero. On the other hand, (γ_1, γ_2) intersects (γ_j, γ_2) only at γ_2. Therefore the pair $([(\gamma_1, \gamma_2)], [(\gamma_j, \gamma_2)])$ is a basis of $H_1(E;\mathbb{Z})$. In the same way, we obtain the fact that the pair $([(\gamma_1, \gamma_2)], [(\gamma_1, \gamma_j)])$ is a basis of $H_1(E;\mathbb{Z})$ in the case (2), and that the pair $([(\gamma_1, \gamma_2)], [\gamma_j])$ is a basis of $H_1(E;\mathbb{Z})$ in the case (3).

Cut E along γ_1, γ_2, and γ_j in each case, and then denote by $\bar{E}$ the resulting Riemann polygon. We obtain a holomorphic map Φ from $\bar{E}$ to $\mathbb{C}P_1$

$$\Phi(q) = \int_{\tilde{q}_0}^{q} \omega_\Lambda$$

when we choose a base point $\tilde{q}_0$ in $\bar{E}$, because we obtain a basis of $H_1(E;\mathbb{Z})$ from γ_1, γ_2, and γ_j.

In the case (1), let $\tilde{q}_0$ be the point in the boundary $\partial\bar{E}$ which is the initial point of γ_1 and is also the end point of γ_2. Then we get the image $\Phi(\bar{E})$ as in the middle figure in Figure 8. It is now easy to modify $\bar{E}$ by cutting and pasting from $\Phi(\bar{E})$ such that we can obtain a Riemann polygon of type II. (See Figure 8.) Note that a' and b' are relatively prime integers where b' is the integral of ω_Λ along (γ_j, γ_2).

In the similar way, we obtain a Riemann polygon of type II in the cases (2) and (3), and omit the explanation.

(ii-b) When the integral of ω_Λ along γ_1 is a real number, there also exists an element γ_j in $\gamma_{(0)}$ different from γ_1 and γ_2. We obtain the fact that γ_j joins q_0 with q_1 as follows: Suppose γ_j is a Λ-edge from q_0 to q_0 itself. We may assume

$$\left(\int_{\gamma_1+\gamma_2} \omega_\Lambda\right) \cdot \left(\int_{\gamma_j} \omega_\Lambda\right) < 0.$$

The simple loop γ_j intersects the simple loop (γ_1, γ_2) at q_0 with intersection number zero because the directions of γ_1, γ_2, and γ_j can be described as in the following figure. Hence it follows from Fact (2) that the homology class $[\gamma_1 + \gamma_2 + \gamma_j]$ vanishes. This contradicts the definition of $\gamma_{(0)}$.

Now we may assume that γ_j is an edge from q_0 to q_1. Then the two homology classes $[(\gamma_1, \gamma_2)]$ and $[(\gamma_j, \gamma_2)]$ form a basis of $H_1(E;\mathbb{Z})$. We can also describe the directions of the three Λ-edges γ_1, γ_2, and γ_j around q_0 and q_1 as in Figure 9.

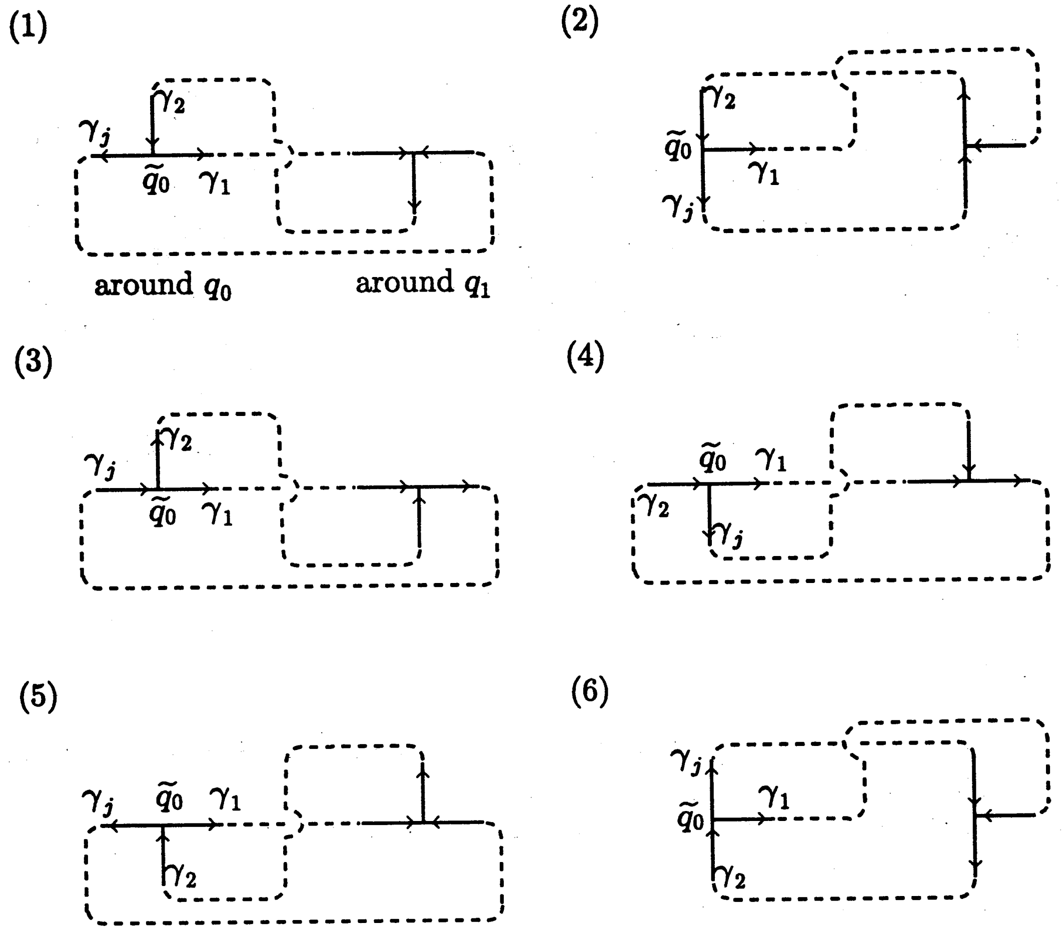


FIGURE 9

Cut E along γ_1, γ_2, and γ_j, and denote by $\bar{E}$ the resulting Riemann polygon. We choose the point $\tilde{q}_0$ at the vertex on $\partial\bar{E}$ where the boundary is smooth. (See Figure 9.) We choose the sign of ω_Λ such that Φ maps each point to a positive number around $\tilde{q}_0$. Then the map Φ gives rise to an isomorphism between (E, p, Λ) and $\widetilde{E}(a, b, x)$ of type III for some (a, b, x). We have completed the proof of Theorem 1. $\square$

We denote by $\mathcal{ML}_{1,1}$ the set of once punctured elliptic curves with Lagrangian sublattices;

$$\mathcal{ML}_{1,1} = \{\widetilde{E}(a, b, x)\}.$$

We will consider the complex structure of $\mathcal{ML}_{1,1}$ in the next section.

3. Complex structure of $\mathcal{ML}_{1,1}$

When we consider the forgetful map from the set $\mathcal{ML}_{1,1}$ of once punctured elliptic curves with Lagrangian sublattices to the moduli space $\mathcal{M}_{1,1}$ of once punctured elliptic curves, it should be a holomorphic map; $\mathcal{ML}_{1,1}$ should be a complex V-manifold whose complex structure is induced from $\mathcal{M}_{1,1}$ by the forgetful map.

We consider the complex structure of $\mathcal{ML}_{1,1}$ by using the description in Theorem 1: We give a local coordinate of $\mathcal{ML}_{1,1}$ around each once punctured elliptic curve with

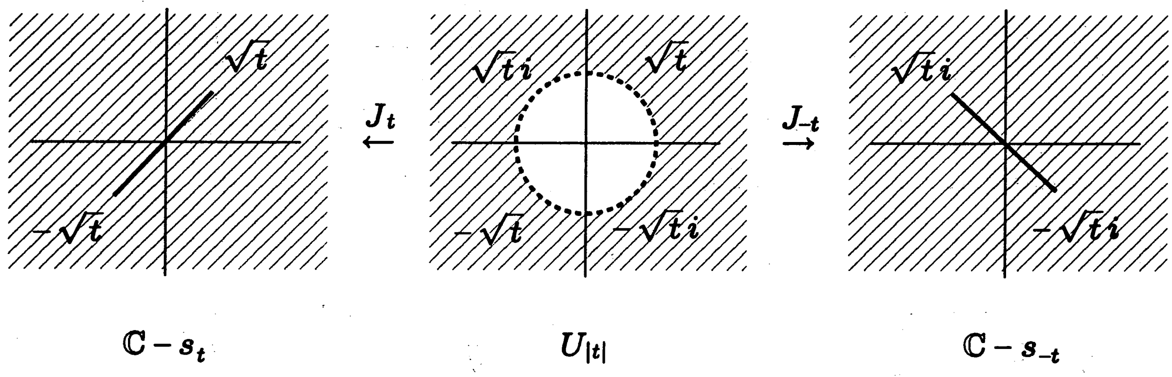


FIGURE 10

a Lagrangian sublattice $\widetilde{E}(a,b,x)$. For this purpose we first prepare two methods of deforming complex structures of Riemann surfaces.

Let t be a complex number and let s_t be the line segment between $\sqrt{t}$ and $-\sqrt{t}$ in $\mathbb{C}$. Set

$$U_{|t|} := \{z \in \mathbb{C}; |z| > |\sqrt{t}|\}.$$

Then the following map J_t is biholomorphic:

$$J_t : U_{|t|} \ni z \longmapsto \frac{1}{2}(z + \frac{t}{z}) \in \mathbb{C} - s_t.$$

Hence we obtain a biholomorphic map $J_t \circ J_{-t}^{-1}$ from $\mathbb{C} - s_{-t}$ to $\mathbb{C} - s_t$. Note that the map $J_t \circ J_{-t}^{-1}$ is parametrized by t holomorphically. (See Figure 10.)

On the other hand, let s'_1 be the union of two line segments; the line segment from 0 to 1 and the line segment from 0 to $e^{\frac{2}{3}\pi i}$. We denote by K the biholomorphic map from $U_1 = \{z \in \mathbb{C}; |z| > 1\}$ to $\mathbb{C} - s'_1$ such that $K(1) = 0$. (There exists uniquely such a biholomorphic map due to Riemann's mapping theorem.) Let t be a complex number as above. We also denote by t the automorphism of $\mathbb{C}$ defined by multiplication by t. Set

$$s'_t = t(s'_1).$$

Then $K_t := t \circ K \circ t^{-1}$ is a biholomorphic map from $U_{|t|}$ to $\mathbb{C} - s'_t$. The map K_t is parametrized by t holomorphically even at $t = 0$. We consider the biholomorphic map $K_t \circ J_{t^2}^{-1}$ from $\mathbb{C} - s_{t^2}$ to $\mathbb{C} - s'_t$, which is also parametrized by t holomorphically. (See Figure 11.)

Let p be a point on a Riemann surface R, and fix a local coordinate $z : \mathcal{U}_p \to \mathbb{C}$ ($z(p) = 0$) around p. The subset $\mathcal{U}_p$ of R is also regarded as a subset of $\mathbb{C}$, and then both sets s_t and s'_t are also subsets of $\mathcal{U}_p$ if t is sufficiently small. We can choose a sufficiently

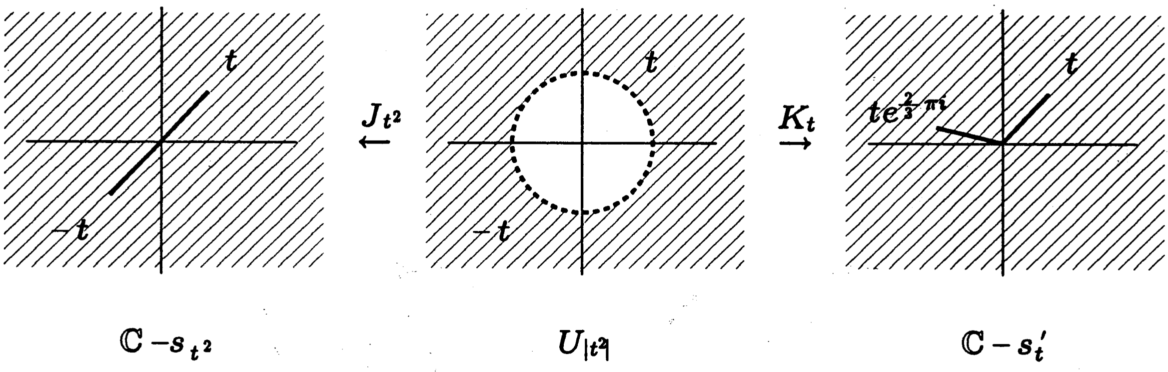


FIGURE 11

small positive number ϵ such that

$$(J_t \circ J_{-t}^{-1})(\{z \in \mathbb{C}; |z| < \epsilon\} - s_{-t}) \subset \mathcal{U}_p,$$
$$(K_t \circ J_{t^2}^{-1})(\{z \in \mathbb{C}; |z| < \epsilon\} - s_{t^2}) \subset \mathcal{U}_p.$$

for any $t \in U_p$.

π-deformation of $(R, (\mathcal{U}_p, z))$: For a complex number $t \in D_\epsilon = \{t \in \mathbb{C}; |t| < \epsilon\}$ we paste $R - s_t$ and $\{z \in \mathbb{C}; |z| < \epsilon\}$ by the attaching map

$$J_t \circ J_{-t}^{-1} : \{z \in \mathbb{C}; |z| < \epsilon\} - s_{-t} \longrightarrow \mathcal{U}_p - s_t \subset R - s_t.$$

Then we obtain a holomorphic family of Riemann surfaces on D_ϵ. Note that the fiber on the origin is R, and that the fiber R_t on t is obtained as follows: cut R along s_t and paste by identifying $k\sqrt{t}$ and $-k\sqrt{t}$ on one side of s_t and identifying $\ell\sqrt{t}$ and $-\ell\sqrt{t}$ on the other side of s_t $(0 < k, \ell \leq 1)$.

$\frac{2}{3}\pi$-deformation of $(R, (\mathcal{U}_p, z))$: For a complex number $t \in D_\epsilon = \{t \in \mathbb{C}; |t| < \epsilon\}$ we paste $R - s'_t$ and $\{z \in \mathbb{C}; |z| < \epsilon\}$ by the attaching map

$$K_t \circ J_{t^2}^{-1} : \{z \in \mathbb{C}; |z| < \epsilon\} - s_{t^2} \longrightarrow \mathcal{U}_p - s'_t \subset R - s'_t.$$

Then we obtain a holomorphic family of Riemann surfaces on D_ϵ. Note that the fiber on the origin is R, and that the fiber R_t on t is obtained as follows: cut R along s'_t and paste by identifying kt and $kte^{\frac{2}{3}\pi i}$ on one side of s'_t and identifying ℓt and $-\ell te^{\frac{2}{3}\pi i}$ on the other side of s'_t $(0 < k, \ell \leq 1)$.

We apply a π-deformation to each $\widetilde{E}(a, b, x)$ of type I, II or III.

Let p_1 be the point on $\widetilde{E}(0, 1, x)$ $(x \in \mathbf{H})$ coming from x and $x + 1$, and fix a local coordinate $z : \mathcal{U}_{p_1} \to \mathbb{C}$ around p_1 such that $\zeta = z^2 + x$ or $\zeta = z^2 + x + 1$, where ζ is the global coordinate of $\mathbb{C}$ as before. When we consider the coordinate $(\mathcal{U}_{p_1}, z)$, the line segment s_t is transferred onto two parallel line segments with the same length t in

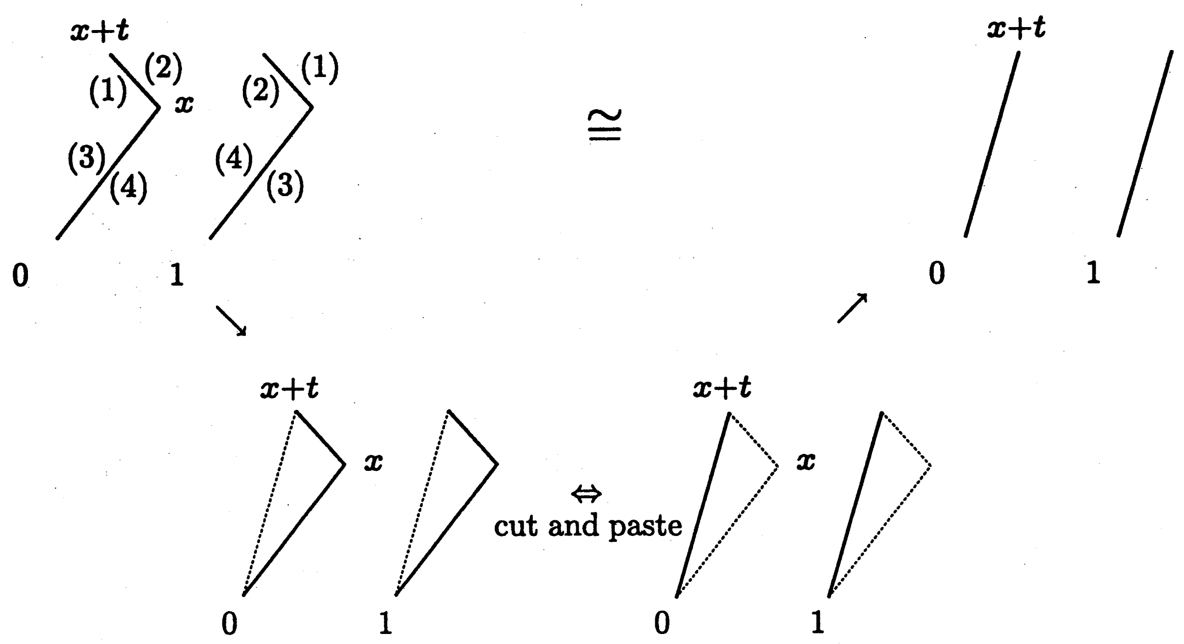


FIGURE 12. π-deformation of $\widetilde{E}(0, 1, x \in \mathbf{H})$

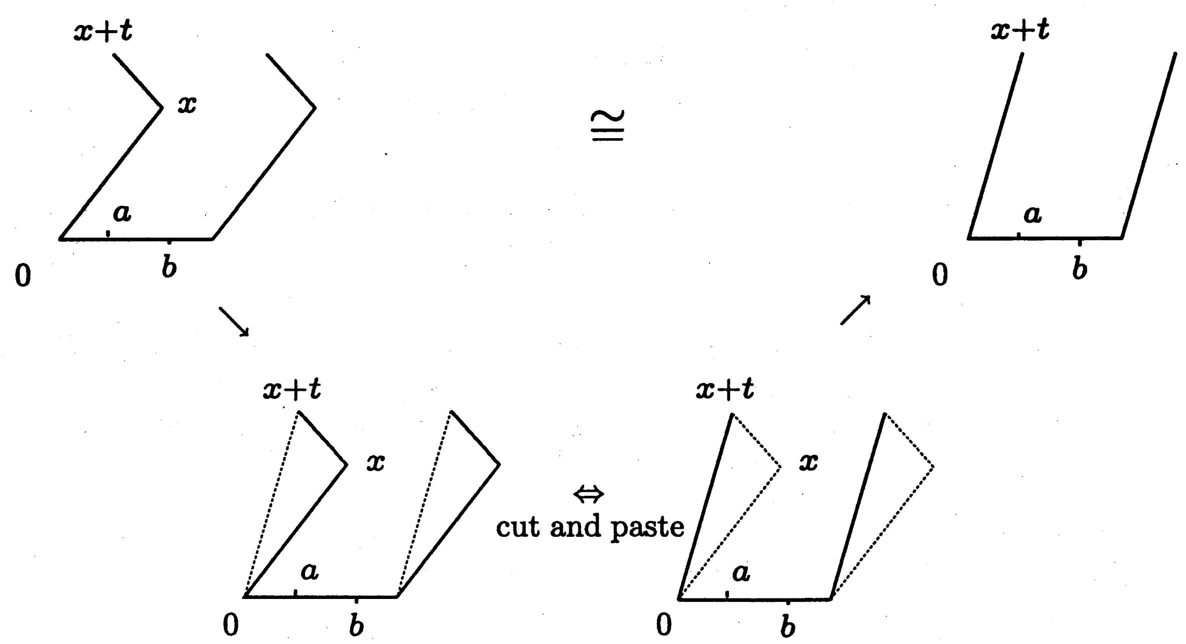


FIGURE 13. π-deformation of $\widetilde{E}(a, b, x)$ of type II

ζ-plane: one is from x to $x + t$ and the other is from $x + 1$ to $x + 1 + t$. The fiber F_t on $t \in D_\epsilon$ is obtained from $\widetilde{E}(0, 1, x)$ by cutting and pasting along the two line segments; F_t is equivalent to $\widetilde{E}(0, 1, x + t)$. (See Figure 12. The numbers (1), ... ,(4) indicate where to paste.) Accordingly we can regard $x \in \mathbf{H}$ as a local coordinate when $a = 0$ and $b = 1$.

Applying a π-deformation to $\widetilde{E}(a, b, x)$ of type II in the same way, we see that x is also a local coordinate if we fix a and b. (See Figure 13.)

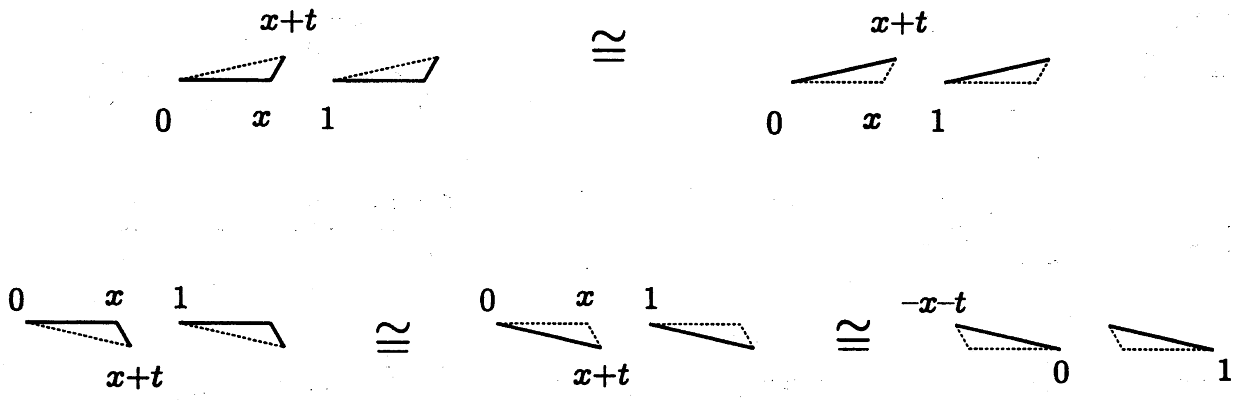


FIGURE 14. π-deformation of $\widetilde{E}(0,1,x\in(0,1))$ of type I

We apply a π-deformation to $\widetilde{E}(0,1,x\in(0,1))$ of type I in the same way, and it follows that the fiber F_t is equivalent to $\widetilde{E}(0,1,x+t)$ when $0\leq\arg t\leq\pi$, and that F_t is equivalent to $\widetilde{E}(0,1,-x-t)$ when $\pi<\arg t<2\pi$ (we use the rotation of angle π). (See Figure 14.)

We also apply a π-deformation to $\widetilde{E}(a,b,x)$ of type III in the same way: Fix a local coordinate $z:\mathcal{U}_{p_1}\to\mathbb{C}$ around p_1 such that

$$\zeta=\begin{cases} z^2+x & -\frac{1}{2}\pi<\arg z<\frac{1}{2}\pi,\\ z^2+x-a & \frac{1}{2}\pi<\arg z<\pi,\\ z^2+x-b & \pi<\arg z<\frac{3}{2}\pi.\end{cases}$$

We consider the case when a is less than b. It is easy to see that the fiber F_t of this deformation is equivalent to $\widetilde{E}(a,b,x+t)$ when $\arg t=0$ or π. When $0<\arg t<\pi$, we see by cutting and pasting that F_t is equivalent to $\widetilde{E}(a,b-a,x-b+t)$. When $\pi<\arg t<2\pi$ and $a\neq1$, we see that F_t is equivalent to $\widetilde{E}(a-(b/a),(b/a),-x+a-t)$ where (b/a) is a non-negative integer such that $0\leq(b/a)<a$ and $(b/a)\equiv b \bmod a$. Note that (b/a) is equal to 0 if and only if a is equal to 1, and then F_t is equivalent to $\widetilde{E}(0,1,-x+1-t)$ of type I. (See Figure 15.) Accordingly we obtain the following injective map $f_{\widetilde{E}(a,b,x)}:D_\epsilon\to\mathcal{ML}_{1,1}$ when a is less than b, and the inverse mapping of $f_{\widetilde{E}(a,b,x)}$ is a local coordinate of $\mathcal{ML}_{1,1}$ around $\widetilde{E}(a,b,x)$.

$$f_{\widetilde{E}(a,b,x)}(t)=\begin{cases}\widetilde{E}(a,b,x+t) & \arg t=0 \text{ or } \pi,\\ \widetilde{E}(a,b-a,x-b+t) & 0<\arg t<\pi,\\ \widetilde{E}(a-(b/a),(b/a),-x+a-t) & \pi<\arg t<2\pi\ (a\neq1),\\ \widetilde{E}(0,1,-x+1-t) & \pi<\arg t<2\pi\ (a=1).\end{cases}$$

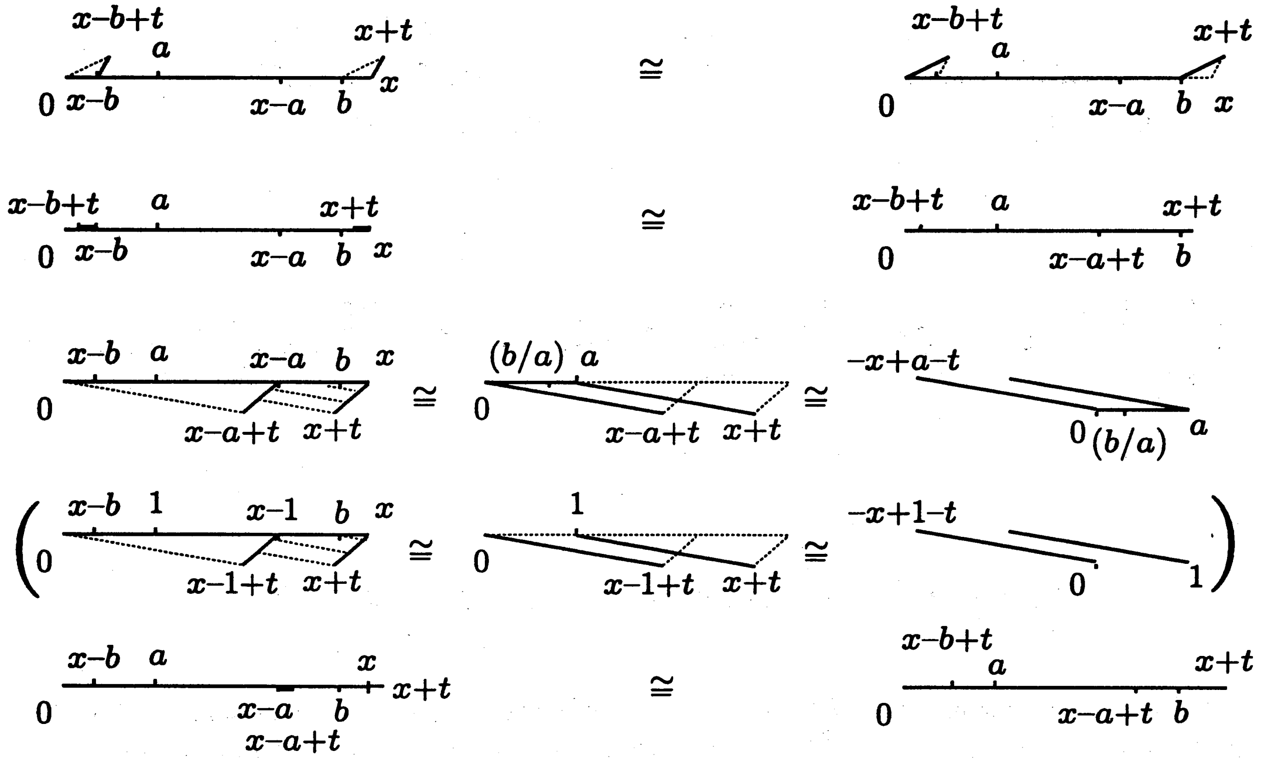


FIGURE 15. π-deformation of $\widetilde{E}(a,b,x)$ of type III ($a<b$)

Similarly we obtain the following injective map $f_{\widetilde{E}(a,b,x)}: D_\epsilon \to \mathcal{ML}_{1,1}$ when a is greater than b, and the inverse mapping of $f_{\widetilde{E}(a,b,x)}$ is a local coordinate of $\mathcal{ML}_{1,1}$ around $\widetilde{E}(a,b,x)$.

$$f_{\widetilde{E}(a,b,x)}(t) = \begin{cases} \widetilde{E}(a,b,x+t) & \arg t = 0 \text{ or } \pi, \\ \widetilde{E}(a-b,b,-x+a-t) & \pi < \arg t < 2\pi, \\ \widetilde{E}((a/b), b-(a/b), x-b+t) & \pi < \arg t < 2\pi. \end{cases}$$

To $\widetilde{E}(a,b,0)$ of type IV we apply a $\frac{2}{3}\pi$-deformation: Fix a local coordinate $z: \mathcal{U}_{p_0} \to \mathbb{C}$ around p_0, the unique zero of ω_0, such that

$$\zeta = \begin{cases} z^3 & 0 < \arg z < \frac{2}{3}\pi, \\ z^3 + a & \frac{2}{3}\pi < \arg z < \pi, \\ z^3 + a + b & \pi < \arg z < \frac{5}{3}\pi, \\ z^3 + b & \frac{5}{3}\pi < \arg z < 2\pi. \end{cases}$$

Then the segment s'_t is transferred to two parallel line segments with the same length t^3 in ζ-plane.

By cutting and pasting we obtain the following map $f_{\widetilde{E}(a,b,0)}: D_\epsilon \to \mathcal{ML}_{1,1}$:

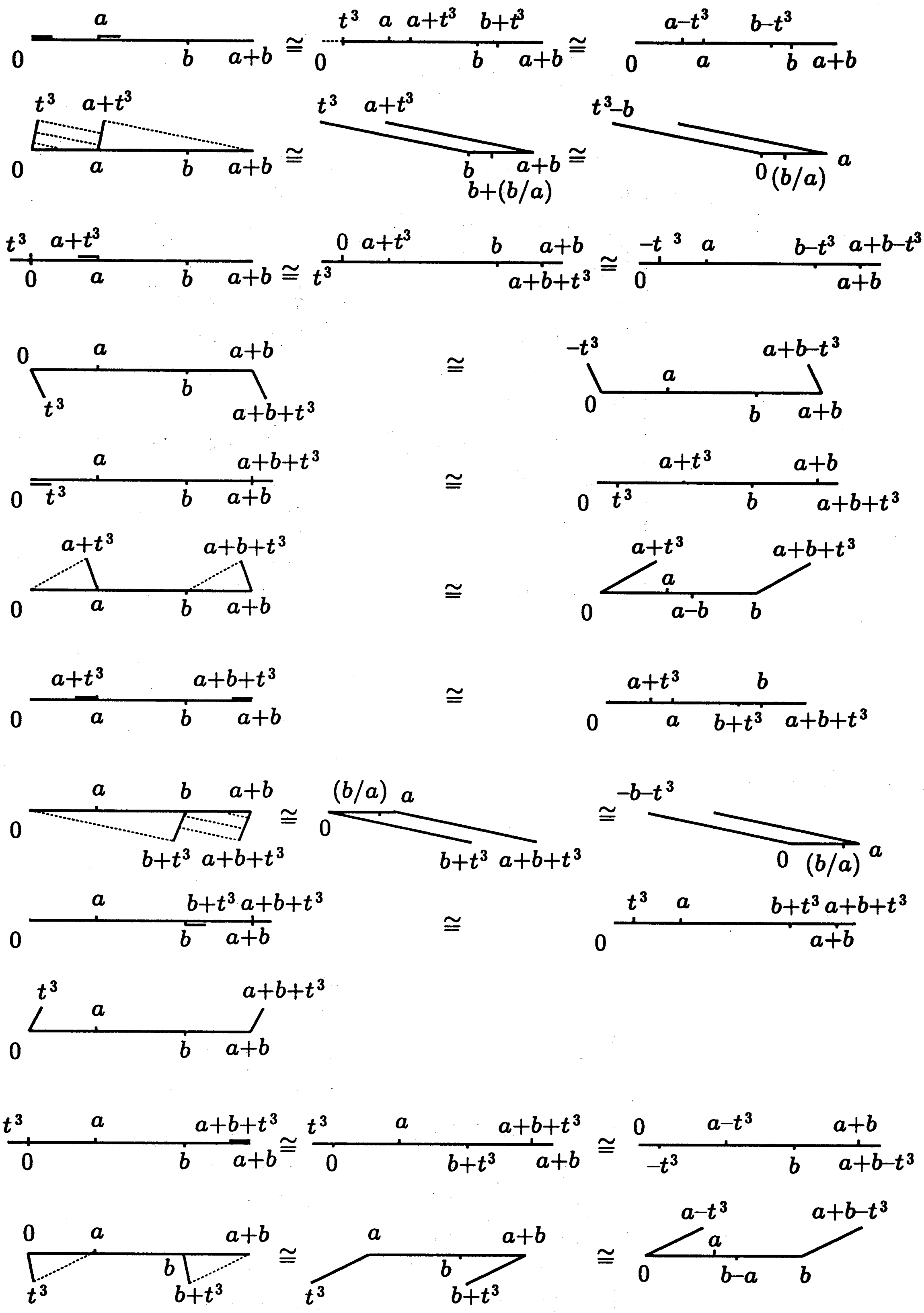


FIGURE 16. $\frac{2}{3}\pi$-deformation of $\widetilde{E}(a, b, 0)$ of type IV ($a < b$ and $a \neq 1$)

When $a < b$ and $a \neq 1$ (Figure 16),

$$f_{\widetilde{E}(a,b,0)}(t) = \begin{cases} \widetilde{E}(a,b,a+b-t^3) & \arg t = 0, \\ \widetilde{E}(a-(b/a),(b/a),t^3-b) & 0 < \arg t < \frac{1}{3}\pi, \\ \widetilde{E}(a,a+b,a+b-t^3) & \arg t = \frac{1}{3}\pi, \\ \widetilde{E}(a,b,-t^3) & \frac{1}{3}\pi < \arg t < \frac{2}{3}\pi, \\ \widetilde{E}(a+b,b,a+b+t^3) & \arg t = \frac{2}{3}\pi, \\ \widetilde{E}(a,b-a,a+t^3) & \frac{2}{3}\pi < \arg t < \pi, \\ \widetilde{E}(a,b,a+b+t^3) & \arg t = \pi, \\ \widetilde{E}(a-(b/a),(b/a),-t^3-b) & \pi < \arg t < \frac{4}{3}\pi, \\ \widetilde{E}(a,a+b,a+b+t^3) & \arg t = \frac{4}{3}\pi, \\ \widetilde{E}(a,b,t^3) & \frac{4}{3}\pi < \arg t < \frac{5}{3}\pi, \\ \widetilde{E}(a+b,b,a+b-t^3) & \arg t = \frac{5}{3}\pi, \\ \widetilde{E}(a,b-a,a-t^3) & \frac{5}{3}\pi < \arg t < 2\pi. \end{cases}$$

When $a = 1$ and $a < b$,

$$f_{\widetilde{E}(1,b,0)}(t) = \begin{cases} \widetilde{E}(1,b,1+b-t^3) & \arg t = 0, \\ \widetilde{E}(0,1,t^3-b) & 0 < \arg t < \frac{1}{3}\pi, \\ \widetilde{E}(1,1+b,1+b-t^3) & \arg t = \frac{1}{3}\pi, \\ \widetilde{E}(1,b,-t^3) & \frac{1}{3}\pi < \arg t < \frac{2}{3}\pi, \\ \widetilde{E}(1+b,b,1+b+t^3) & \arg t = \frac{2}{3}\pi, \\ \widetilde{E}(1,b-1,1+t^3) & \frac{2}{3}\pi < \arg t < \pi, \\ \widetilde{E}(1,b,1+b+t^3) & \arg t = \pi, \\ \widetilde{E}(0,1,-t^3-b) & \pi < \arg t < \frac{4}{3}\pi, \\ \widetilde{E}(1,1+b,1+b+t^3) & \arg t = \frac{4}{3}\pi, \\ \widetilde{E}(1,b,t^3) & \frac{4}{3}\pi < \arg t < \frac{5}{3}\pi, \\ \widetilde{E}(1+b,b,1+b-t^3) & \arg t = \frac{5}{3}\pi, \\ \widetilde{E}(1,b-1,1-t^3) & \frac{5}{3}\pi < \arg t < 2\pi. \end{cases}$$

When $a = b = 1$,

$$f_{\widetilde{E}(1,1,0)}(t) = \begin{cases} \widetilde{E}(0,1,1-t^3) & \arg t = 0, \\ \widetilde{E}(0,1,t^3-1) & 0 < \arg t < \frac{1}{3}\pi, \\ \widetilde{E}(1,2,2-t^3) & \arg t = \frac{1}{3}\pi, \\ \widetilde{E}(1,1,-t^3) & \frac{1}{3}\pi < \arg t < \frac{2}{3}\pi, \\ \widetilde{E}(2,1,2+t^3) & \arg t = \frac{2}{3}\pi, \\ \widetilde{E}(0,1,1+t^3) & \frac{2}{3}\pi < \arg t < \pi, \\ \widetilde{E}(0,1,1+t^3) & \arg t = \pi, \\ \widetilde{E}(0,1,-t^3-1) & \pi < \arg t < \frac{4}{3}\pi, \\ \widetilde{E}(1,2,2+t^3) & \arg t = \frac{4}{3}\pi, \\ \widetilde{E}(1,1,t^3) & \frac{4}{3}\pi < \arg t < \frac{5}{3}\pi, \\ \widetilde{E}(2,1,2-t^3) & \arg t = \frac{5}{3}\pi, \\ \widetilde{E}(0,1,1-t^3) & \frac{5}{3}\pi < \arg t < 2\pi. \end{cases}$$

When $b = 1$ and $a > b$,

$$f_{\widetilde{E}(a,1,0)}(t) = \begin{cases} \widetilde{E}(a,1,a+1-t^3) & \arg t = 0, \\ \widetilde{E}(a-1,1,t^3-1) & 0 < \arg t < \frac{1}{3}\pi, \\ \widetilde{E}(a,a+1,a+1-t^3) & \arg t = \frac{1}{3}\pi, \\ \widetilde{E}(a,1,-t^3) & \frac{1}{3}\pi < \arg t < \frac{2}{3}\pi, \\ \widetilde{E}(a+1,1,a+1+t^3) & \arg t = \frac{2}{3}\pi, \\ \widetilde{E}(0,1,a+t^3) & \frac{2}{3}\pi < \arg t < \pi, \\ \widetilde{E}(a,1,a+1+t^3) & \arg t = \pi, \\ \widetilde{E}(a-1,1,-t^3-1) & \pi < \arg t < \frac{4}{3}\pi, \\ \widetilde{E}(a,a+1,a+1+t^3) & \arg t = \frac{4}{3}\pi, \\ \widetilde{E}(a,1,t^3) & \frac{4}{3}\pi < \arg t < \frac{5}{3}\pi, \\ \widetilde{E}(a+1,1,a+1-t^3) & \arg t = \frac{5}{3}\pi, \\ \widetilde{E}(0,1,a-t^3) & \frac{5}{3}\pi < \arg t < 2\pi. \end{cases}$$

When $b \neq 1$ and $a > b$,

$$f_{\widetilde{E}(a,b,0)}(t) = \begin{cases} \widetilde{E}(a,b,a+b-t^3) & \arg t = 0, \\ \widetilde{E}(a-b,b,t^3-b) & 0 < \arg t < \frac{1}{3}\pi, \\ \widetilde{E}(a,a+b,a+b-t^3) & \arg t = \frac{1}{3}\pi, \\ \widetilde{E}(a,b,-t^3) & \frac{1}{3}\pi < \arg t < \frac{2}{3}\pi, \\ \widetilde{E}(a+b,b,a+b+t^3) & \arg t = \frac{2}{3}\pi, \\ \widetilde{E}((a/b),b-(a/b),a+t^3) & \frac{2}{3}\pi < \arg t < \pi, \\ \widetilde{E}(a,b,a+b+t^3) & \arg t = \pi, \\ \widetilde{E}(a-b,b,-t^3-b) & \pi < \arg t < \frac{4}{3}\pi, \\ \widetilde{E}(a,a+b,a+b+t^3) & \arg t = \frac{4}{3}\pi, \\ \widetilde{E}(a,b,t^3) & \frac{4}{3}\pi < \arg t < \frac{5}{3}\pi, \\ \widetilde{E}(a+b,b,a+b-t^3) & \arg t = \frac{5}{3}\pi, \\ \widetilde{E}((a/b),b-(a/b),a-t^3) & \frac{5}{3}\pi < \arg t < 2\pi. \end{cases}$$

For each $\widetilde{E}(a,b,0)$ of type IV, the map $f_{\widetilde{E}(a,b,0)}$ gives rise to a two-fold branched covering;

$$f_{\widetilde{E}(a,b,0)}(t) = f_{\widetilde{E}(a,b,0)}(-t).$$

Then $(f_{\widetilde{E}(a,b,0)}, D_\epsilon, \mathbb{Z}/2\mathbb{Z})$ gives rise to a local manifold cover of $\mathcal{ML}_{1,1}$ around $\widetilde{E}(a,b,0)$. In this way, the set $\mathcal{ML}_{1,1}$ is regarded as a complex V-manifold.

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